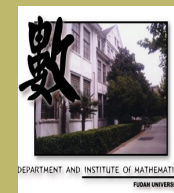


12,19,2006 · HangZhou



Initial-boundary value problem for the equation of time-like extremal surfaces in Minkowski space

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Joint work with Liu Jianli

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We denote $(x_0, x_1, \dots, x_{n+1})$ a point in the $(n + 1) + 1$ dimensional Minkowski space endorsed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + \dots + dx_{n+1}^2$$

Let

$$x_0 = t, x_1 = x, x_2 = \phi_1(t, x), \dots, x_{n+1} = \phi_n(t, x)$$

be a two dimensional time like surface. Then the induced metric on the surface is

$$\begin{aligned} d_*s^2 &= -dt^2 + dx^2 + d(\phi_1)^2 + \dots + d(\phi_n)^2 \\ &= -(1 - (\phi_t)^2)dt^2 + (1 + (\phi_x)^2)dx^2 + 2\phi_t \cdot \phi_x dxdt \end{aligned}$$

where $\phi = (\phi_1, \dots, \phi_n)^T$, ϕ_t or ϕ_x denote partial differentiation with respect to t or x respectively and \cdot denotes inner product in R^n . Thus, it is easy to see that the area of the surface is

$$\int \int \sqrt{1 - (\phi_t)^2 + (\phi_x)^2 - (\phi_t)^2(\phi_x)^2 + (\phi_t \cdot \phi_x)^2} dxdt$$

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An extremal surface is defined to be the extremal point of the area functional, hence it satisfies the Euler-Lagrange equations

$$\begin{aligned} & \left(\frac{(1 + (\phi_x)^2)\phi_t - (\phi_t \cdot \phi_x)\phi_x}{\sqrt{1 - (\phi_t)^2 + (\phi_x)^2 - (\phi_t)^2(\phi_x)^2 + (\phi_t \cdot \phi_x)^2}} \right)_t \\ & - \left(\frac{(1 - (\phi_t)^2)\phi_x + (\phi_t \cdot \phi_x)\phi_t}{\sqrt{1 - (\phi_t)^2 + (\phi_x)^2 - (\phi_t)^2(\phi_x)^2 + (\phi_t \cdot \phi_x)^2}} \right)_x = 0 \end{aligned} \quad (4)$$

We consider the Cauchy problem for system (4) with initial data

$$\phi(0, x) = h(x), \quad \phi_t(0, x) = g(x)$$

where h' and g are vector valued C^1 functions.

Let

$$u = \phi_x, \quad v = \phi_t$$

Then, Eq.(4) can be equivalently rewritten as a first order systems of conservation laws for the unknown $U(t, x) = (u(t, x), v(t, x))$ as follows

$$u_t - v_x = 0$$

$$\left(\frac{(1 + u^2)v - (u \cdot v)u}{\sqrt{1 - v^2 + u^2 - v^2u^2 + (u \cdot v)^2}} \right)_t - \left(\frac{(1 - v^2)u + (u \cdot v)v}{\sqrt{1 - v^2 + u^2 - v^2u^2 + (u \cdot v)^2}} \right)_x = 0$$

The initial condition then becomes $(u(0, x), v(0, x)) = U_0(x) = (h'(x), g(x))$. This is an interesting model in Lorentian geometry.

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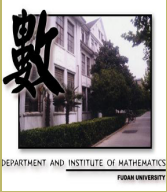
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◆ Barbashov, Nesterenko and Chervyakov [10](Commun.Math.Phys.(1982))
Milnor [11] (Michigan Math. (1990))

◆ Gu [12] (Nonlinear Differential Equations Appl 4. (1990)). [13] (Chinese
Ann.Math (1994))
Kong [14] (Europhys.Lett. (2004))

◆ D. Chae and H. Huh [15] (J. Math. Phys(2004))
H.Lindblad [16] (Proc. Am. Math. Soc (2004))
Kong, Sun and Zhou [17] (J.Math.phys (2006))

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In [17] they showed that system can be reduced to

$$\begin{cases} u_t - v_x = 0 \\ v_t - \frac{2(u \cdot v)}{1 + u^2} v_x - \frac{1 - v^2}{1 + u^2} u_x = 0 \end{cases}$$

They found that it enjoys many interesting properties: nonstrictly hyperbolicity, constant multiplicity of eigenvalues, linear degeneracy of all characteristic fields, richness, etc. The system have two n-constant multiple eigenvalues:

$$\lambda_{\pm} = \frac{1}{1 + u^2} (-(u \cdot v) \pm \sqrt{\Delta(u, v)})$$

where $\Delta(u, v) = 1 - v^2 + u^2 - u^2 v^2 + (u \cdot v)^2 > 0$. They also proved

$$|\lambda_{\pm}(t, x)| \leq 1$$

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Let

$$R_i = v_i + \lambda_+ u_i, \quad (i = 1, \dots, n)$$

$$S_i = v_i + \lambda_- u_i, \quad (i = 1, \dots, n)$$

then they satisfies the following systems

$$\begin{cases} \partial_t \lambda_+ + \lambda_- \partial_x \lambda_+ = 0 \\ \partial_t R_i + \lambda_- \partial_x R_i = 0 \quad (i = 1, \dots, n) \\ \partial_t \lambda_- + \lambda_+ \partial_x \lambda_- = 0 \\ \partial_t S_i + \lambda_+ \partial_x S_i = 0 \quad (i = 1, \dots, n) \end{cases}$$

$$t = 0 : \lambda_+(0, x) = \Lambda_+(x), \lambda_-(0, x) = \Lambda_-(x),$$

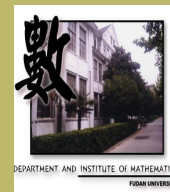
$$R_i(0, x) = R_i^0(x), \quad S_i(0, x) = S_i^0(x)$$

Then it was proved that above system admits a global classical solution for all $t \in R^+$, provided that U_0 is C^1 and the strictly hyperbolic condition

$$\delta = \inf_{x \in R} \Lambda_+(x) - \sup_{x \in R} \Lambda_-(x) > 0 \quad (5)$$

is satisfied.

$$u_i(t, x) = \frac{R_i(t, x) - S_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)}, \quad v_i(t, x) = \frac{\lambda_+ S_i(t, x) - \lambda_- R_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)}$$



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Theorem A. Suppose that (5) is satisfied, then the cauchy problem (4) admits a unique global C^2 solution $\phi = \phi(t, x)$ on $R^+ \times R$. Moreover, it holds that

$$\Delta(\phi_x(t, x), \phi_t(t, x)) > 0, \quad \forall (t, x) \in R^+ \times R$$

Under the following assumptions:

$$\sup_{x \in R} \{|h''(x)| + |g'(x)|\} \doteq \bar{M} < \infty,$$

$$\int_{-\infty}^{+\infty} |h'(x)| + |g(x)| dx \doteq \bar{N}_1 < \infty$$

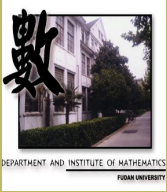
$$\int_{-\infty}^{+\infty} |h''(x)| + |g'(x)| dx \doteq \bar{N}_2 < \infty$$

$$\bar{M}_0 = \sup_{x \in R} \{|h'(x)| + |g(x)|\}$$

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Firstly we consider the Cauchy problem:

- Consider Cauchy problem of the first order quasilinear hyperbolic systems

$$\begin{aligned}\frac{\partial u}{\partial t} + A(u) \frac{\partial u}{\partial x} &= B(u), \\ u(0, x) &= f(x)\end{aligned}$$

◆ The existence of the global classical solutions:

Bressan [1] Indiana University Mathematics Journal (1988)

Li [2] (published in the United States with John Wiley & Sons, 1994.)

Li, Zhou and Kong [3] Comm.PDE (1994). [4] Nonl.Anal.(1997)

Kong [5] ,

Zhou [6] Chin. Ann. Math. (2004)

◆ Asymptotic behavior:

Kong and Yang [7] Comm in Part Diff Eqs. (2003)

Dai and Kong [8] Chin. Ann. Math. B (2006), [9] (preprint).

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★★In the following we consider the following diagonalizable quasilinear hyperbolic systems

$$\frac{\partial u_i}{\partial t} + \lambda_i(u) \frac{\partial u_i}{\partial x} = 0 \quad (1)$$

where $u = (u_1, \dots, u_n)^T$ is the unknown vector-valued function of (t, x) . $\lambda_i(u)$ is given C^2 vector-valued function of u and is **linearly degenerate**, i.e.

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0$$

and the system (1) is **strictly hyperbolic**, i.e.

$$\lambda_1(u) < \dots < \lambda_n(u)$$

Suppose that there exists a positive constant δ such that,

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta, \quad i = 1, \dots, n - 1.$$

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Consider the cauchy problem for the system (1) with the following initial data

$$t = 0 : \quad u = f(x)$$

where $f(x)$ is a C^1 vector-valued function of x . The global existence of the classical solutions is well-known see Li [2].

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Assumptions

$$\sup_{x \in R} |f'(x)| \doteq M < \infty$$

$$\int_{-\infty}^{+\infty} |f(x)| dx \doteq N_1 < \infty$$

$$\int_{-\infty}^{+\infty} |f'(x)| dx \doteq N_2 < \infty$$

$$\sup_{x \in R} |f(x)| \doteq M_0$$

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Theorem 1.1. Under the assumptions of above, there exists a unique C^1 vector-valued function $\phi(x) = (\phi_1(x), \dots, \phi_n(x))^T$ such that,

$$u(t, x) \longrightarrow \sum_{i=1}^n \phi_i(x - \lambda_i(0)t) e_i, \quad t \longrightarrow \infty$$

where

$$e_i = (0, \dots, 1^i, 0, \dots, 0)^T$$

Moreover, $\phi_i(x) (i = 1, \dots, n)$ is **global Lipschitz continuous**. Furthermore, If system (1) is rich and the derivative of the initial data $f'(x)$ is **global ρ -holder continuous**, where $0 < \rho \leq 1$, i.e. there exists a positive constant κ independent of $\alpha, \beta \in R$ such that,

$$|f'(\alpha) - f'(\beta)| \leq \kappa |\alpha - \beta|^\rho$$

Then $\phi'_i(x) (i = 1, \dots, n)$ satisfies

$$|\phi'_i(\alpha) - \phi'_i(\beta)| \leq C \tilde{M}(|\alpha - \beta|^\rho + |\alpha - \beta|)$$

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Lemma 4.1. Under the assumptions above, the limit

$$\lambda_-(t, x) \longrightarrow \tilde{\psi}(x - t), \quad t \longrightarrow \infty$$

exists and for any $\alpha, \beta \in R$, we have

$$|\tilde{\psi}(\alpha) - \tilde{\psi}(\beta)| \leq C\bar{M}|\alpha - \beta|$$

Moreover,

$$|\tilde{\psi}(\alpha)| \leq 1$$

Remark 4.1. Using the similar method, we can get the following estimate

$$S_i(t, x) \longrightarrow \tilde{\phi}_i(x - t), \text{ exist and } |\tilde{\phi}_i(\alpha) - \tilde{\phi}_i(\beta)| \leq C\bar{M}|\alpha - \beta|$$

$$\lambda_+(t, x) \longrightarrow \psi(x + t), \text{ exist and } |\psi(\alpha) - \psi(\beta)| \leq C\bar{M}|\alpha - \beta|$$

$$R_i(t, x) \longrightarrow \check{\phi}_i(x + t), \text{ exist and } |\check{\phi}_i(\alpha) - \check{\phi}_i(\beta)| \leq C\bar{M}|\alpha - \beta|$$

Moreover,

$$|\psi(\alpha)| \leq 1, |\tilde{\phi}_i(\alpha)|, |\check{\phi}_i(\alpha)| \leq 1 + C\bar{M}\bar{N}_1$$

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Then

$$\lim_{t \rightarrow \infty} u_i(t, x) = \lim_{t \rightarrow \infty} \frac{\check{\phi}_i(x+t) - \tilde{\phi}_i(x-t)}{\psi(x+t) - \tilde{\psi}(x-t)}$$

$$\lim_{t \rightarrow \infty} v_i(t, x) = \lim_{t \rightarrow \infty} \frac{\psi(x+t)\tilde{\phi}_i(x-t) - \tilde{\psi}(x-t)\check{\phi}_i(x+t)}{\psi(x+t) - \tilde{\psi}(x-t)}$$

Then, we can get

when $x \geq 0$

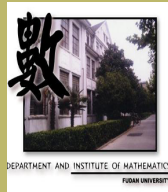
$$\lim_{t \rightarrow \infty} u_i(t, x) = \frac{-\tilde{\phi}_i(x-t)}{1 - \tilde{\psi}(x-t)} \doteq \Phi_i(x-t), \quad i = 1, \dots, n$$

$$\lim_{t \rightarrow \infty} v_i(t, x) = \frac{\tilde{\phi}_i(x-t)}{1 - \tilde{\psi}(x-t)} \doteq -\Phi_i(x-t), \quad i = 1, \dots, n$$

when $x \leq 0$

$$\lim_{t \rightarrow \infty} u_i(t, x) = \frac{\check{\phi}_i(x+t)}{1 + \psi(x+t)} \doteq \Psi_i(x+t), \quad i = 1, \dots, n$$

$$\lim_{t \rightarrow \infty} v_i(t, x) = \frac{\check{\phi}_i(x+t)}{1 + \psi(x+t)} \doteq \Psi_i(x+t), \quad i = 1, \dots, n$$



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So we can conclude

$$u_i(t, x) \longrightarrow \Phi_i(x - t) + \Psi_i(x + t)$$

$$v_i(t, x) \longrightarrow -\Phi_i(x - t) + \Psi_i(x + t)$$

We also have conclusion that $\Phi_i(\alpha)$ and $\Psi_i(\alpha)$ ($i = 1, \dots, n$) are Lipschitz continuous.

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Theorem 1.2. Under the assumptions of above, furthermore, suppose initial data satisfies (5). There exist unique C^1 vector-valued functions $\Phi(x) = (\Phi_1(x), \dots, \Phi_n(x))$ and $\Psi(x) = (\Psi_1(x), \dots, \Psi_n(x))$ such that,

$$(\phi_i)_x \longrightarrow \Phi_i(x - t) + \Psi_i(x + t), \quad t \longrightarrow \infty \quad i = 1, \dots, n$$

$$(\phi_i)_t \longrightarrow -\Phi_i(x - t) + \Psi_i(x + t) \quad t \longrightarrow \infty \quad i = 1, \dots, n$$

Moreover $\Phi_i(x)$ and $\Psi_i(x)$ ($i = 1, \dots, 2n$) are global Lipschitz continuous. Exactly, there exists a positive constant \tilde{M} only depending on $\bar{M}, \bar{N}_1, \bar{N}_2$. It holds that

$$|\Phi_i(\alpha) - \Phi_i(\beta)| \leq C\tilde{M}|\alpha - \beta|$$

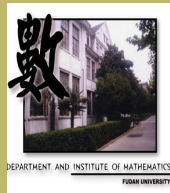
$$|\Psi_i(\alpha) - \Psi_i(\beta)| \leq C\tilde{M}|\alpha - \beta|$$

Furthermore, If the initial data h'', g' are global ρ -hölder ($0 < \rho \leq 1$) continuous, i.e. there exists a positive constant κ independent of $\alpha, \beta \in R$ such that,

$$|h''(\alpha) - h''(\beta)| + |g'(\alpha) - g'(\beta)| \leq \kappa|\alpha - \beta|^\rho$$

then, $\Phi'_i(x)$ and $\Psi'_i(x)$ satisfy

$$|\Phi'_i(\alpha) - \Phi'_i(\beta)| + |\Psi'_i(\alpha) - \Psi'_i(\beta)| \leq C\kappa\tilde{M}^\rho|\alpha - \beta|^\rho + C\tilde{M}|\alpha - \beta|$$


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★★Initial-boundary value problem for the equation of time-like extremal surface in Minkowski space

We only consider the initial-boundary value problem with Neumann boundary condition

$$\begin{cases} u_t - v_x = 0 \\ v_t - \frac{2(u \cdot v)}{1 + u^2}v_x - \frac{1 - v^2}{1 + u^2}u_x = 0 \\ t = 0 : u = f'(x), v = g(x) \\ x = 0 : u = h(t) \end{cases} \quad (1.16)$$

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Suppose that U_0, h are C^1 functions with bounded C^1 norm and the initial conditions satisfy

$$\sup_{x \in R^+} \Lambda_-(x) \leq -a < 0 < b \leq \inf_{x \in R^+} \Lambda_+(x) \quad (1.25)$$

Without loss of generality, we assume $a < b$. (Otherwise, we can always replace a by a smaller positive number). If the Neumann boundary data is sufficiently small, for example

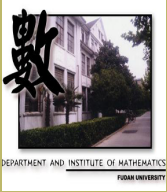
$$|h(t)| \leq \frac{b-a}{3} \quad (1.26)$$

Then we have the following global existence result for the initial-boundary value problem (1.5)-(1.7)

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Theorem 1.1 Suppose that the initial data and Neumann boundary data satisfy (1.25), (1.26), the conditions of C^2 compatibility (1.9) are satisfied, then the initial-boundary value problem (1.5)-(1.7) admits a unique global C^2 solution $\phi = \phi(t, x)$ on $R^+ \times R^+$.

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Similarly, we suppose U_0, H' are C^1 functions with bounded C^1 norm and the initial conditions satisfy

$$\sup_{x \in R^+} \Lambda_-(x) \leq -a < 0 < b \leq \inf_{x \in R^+} \Lambda_+(x)$$

Without loss of generality, we assume $a < b$. If the first derivative of Dirichlet boundary data is sufficiently small, for example

$$|H'(t)| \leq b - a \quad (1.27)$$

The conditions of C^2 compatibility are satisfied. i.e.

$$f(0) = H(0), \quad H'(0) = g(0) \quad (1.28)$$

and

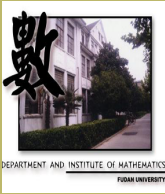
$$H''(0) - \frac{2f'(0) \cdot g(0)}{1 + f'^2(0)} g'(0) - \frac{1 - g^2(0)}{1 + f'^2(0)} f''(0) = 0 \quad (1.29)$$

Then we have the following global existence result

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Theorem 1.2 Suppose the above assumptions (1.25) and (1.27)-(1.29) are satisfied, then the initial-boundary value problem (1.5), (1.6) and (1.8) admits a unique global C^2 solution $\phi = \phi(t, x)$ on $R^+ \times R^+$.

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If we also suppose that the initial and boundary datum satisfy the following assumptions:

$$\sup_{x \in R^+} \{|f''(x)| + |g'(x)|\} \doteq N < \infty \quad (1.30)$$

$$\int_0^{+\infty} |f'(x)| + |g(x)| dx \doteq N_1 < \infty \quad (1.31)$$

$$\int_0^{+\infty} |f''(x)| + |g'(x)| dx \doteq N_2 < \infty \quad (1.32)$$

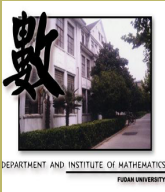
$$\sup_{x \in R^+} \{|f'(x)| + |g(x)|\} = N_0 \quad (1.33)$$

$$\sup_{t \in R^+} \{|h'(t)|\} \doteq M < \infty \quad (1.34)$$

$$\int_0^{+\infty} |h(t)| dt \doteq M_1 < \infty \quad (1.35)$$

$$\int_0^{+\infty} |h'(t)| dt \doteq M_2 < \infty \quad (1.36)$$

Based on the existence of the global classical solutions, we also prove the following Theorem:


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Theorem 1.3 Under the assumptions of Theorem 1.1 and above, there exists a unique C^1 vector-valued function $\Phi(x) = (\Phi_1(x), \dots, \Phi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Phi_i(x - t), -\Phi_i(x - t)) \quad i = 1, \dots, n \quad (1.37)$$

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Similarly, under the assumptions that

$$\sup_{x \in R^+} \{|f''(x)| + |g'(x)|\} \doteq N < \infty$$

$$\int_0^{+\infty} |f'(x)| + |g(x)| dx \doteq N_1 < \infty$$

$$\int_0^{+\infty} |f''(x)| + |g'(x)| dx \doteq N_2 < \infty$$

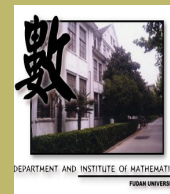
$$\sup_{x \in R^+} \{|f'(x)| + |g(x)|\} = N_0$$

$$\sup_{t \in R^+} \{|H''(t)|\} \doteq M < \infty \quad (1.38)$$

$$\int_0^{+\infty} |H'(t)| dt \doteq M_1 < \infty \quad (1.39)$$

$$\int_0^{+\infty} |H''(t)| dt \doteq M_2 < \infty \quad (1.40)$$

We have the following Theorem


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Theorem 1.4 Under the assumptions of Theorem 1.2 and above, there exists a unique C^1 vector-valued function $\Psi(x) = (\Psi_1(x), \dots, \Psi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Psi_i(x - t), -\Psi_i(x - t)) \quad i = 1, \dots, n \quad (1.41)$$

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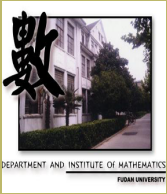
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★ **Key idea of the proof**

Lemma 2.1 Under the assumptions of (1.9), (1.25) and (1.26), system (1.20) is strictly hyperbolicity. Furthermore, on the domain D we have

$$\lambda_-(t, x) \leq -a < 0 < b \leq \lambda_+(t, x) \quad (2.3)$$

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Proof: For any fixed $(t, x) \in D$, we draw the forward characteristic $\tilde{C}_1 : x = x_1(t)$ through this point. There are only two possibilities:

Case 1: The forward characteristic $\tilde{C}_1 : x = x_1(t, \beta)$ intersects x axis at a point $(0, \beta)$. By the second equation of system (2.1), $\lambda_-(t, x)$ is a constant along any given forward characteristic, then we have

$$\lambda_-(t, x) = \Lambda_-(\beta) \quad (2.4)$$

Noting (1.25)

$$\lambda_-(t, x) \leq -a \quad \forall (t, x) \in D \quad (2.5)$$

In the similar way, we draw the backward characteristic $\tilde{C}_2 : x = x_2(t, \alpha)$ intersects x axis at a point $(0, \alpha)$. Along any given backward characteristic $\lambda_+(t, x)$ is a constant

$$\lambda_+(t, x) = \Lambda_+(\alpha) \quad (2.6)$$

Then

$$b \leq \lambda_+(t, x) \quad \forall (t, x) \in D \quad (2.7)$$

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Case 2: The forward characteristic $\tilde{C}_1 : x = x_1(t)$ intersects t axis at a point $(\gamma, 0)$ and the backward characteristic $\tilde{C}_2 : x = x_2(t)$ passing through $(\gamma, 0)$ intersects x axis at a point $(0, \alpha)$, where $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\frac{dx_2(t)}{dt} = \lambda_-(t, x_2(t, \alpha)), \quad x_2(0, \alpha) = \alpha$$

Similarly, we can get

$$\lambda_-(t, x) = \lambda_-(\gamma, 0) \quad (2.8)$$

$$\lambda_+(\gamma, 0) = \Lambda_+(\alpha) \quad (2.9)$$

Then

$$\lambda_-(t, x) + \Lambda_+(\alpha) = \frac{-2(h \cdot v)}{1 + |h|^2} \quad (2.12)$$

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Let $\theta \in [0, \pi]$ is the angle between vectors h and v . We have $h \cdot v = |h||v| \cos \theta$. Then, Equation (2.10) can be rewritten as

$$(1 + |h|^2)\Lambda_+(\alpha) + |h||v| \cos \theta = \sqrt{1 - |v|^2 + |h|^2 - |v|^2|h|^2 + |h|^2|v|^2 \cos^2 \theta} \quad (2.13)$$

Then

$$(1 + |h|^2)^2 \Lambda_+^2(\alpha) + (1 + |h|^2)|v|^2 + 2(1 + |h|^2)\Lambda_+(\alpha)|h||v| \cos \theta = 1 + |h|^2 \quad (2.14)$$

Dividing (2.14) by $1 + |h|^2$ leads to

$$(1 + |h|^2)\Lambda_+^2(\alpha) + |v|^2 + 2\Lambda_+(\alpha)|h||v| \cos \theta = 1 \quad (2.15)$$

Noting $\theta \in [0, \pi]$, then $|\cos \theta| \leq 1$

$$(1 + |h|^2)\Lambda_+^2(\alpha) + |v|^2 - 2\Lambda_+(\alpha)|h||v| \leq 1$$

$$(|v| - |h|\Lambda_+(\alpha))^2 \leq 1 - \Lambda_+^2(\alpha)$$

Notice that the initial data satisfies (1.25), we can get

$$|v| \leq \Lambda_+(\alpha)|h| + \sqrt{1 - \Lambda_+^2(\alpha)} \quad (2.16)$$

Thus

$$|v| \leq |h| + 1 \quad (2.17)$$

By Equation (2.12), we have

$$\begin{aligned} -\lambda_-(t, x) &= \Lambda_+(\alpha) + \frac{2|h||v| \cos \theta}{1 + |h|^2} \\ &\geq \Lambda_+(\alpha) - \frac{2|h||v|}{1 + |h|^2} \end{aligned} \quad (2.18)$$

Noting (1.25) and (2.17)

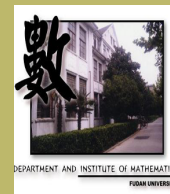
$$-\lambda_-(t, x) \geq b - \frac{2|h|(|h| + 1)}{1 + |h|^2} \quad (2.19)$$

On the other hand, the Neumann boundary data satisfies (1.26), then

$$\frac{2|h|(|h| + 1)}{1 + |h|^2} \leq b - a$$

Therefore

$$\lambda_-(t, x) \leq -a \quad (2.20)$$



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Lemma 2.2 Let R_i, S_i be as system (1.20), then

$$\{|R_i(t, x)|, |S_i(t, x)|\} \leq C \quad (2.21)$$

where C is a positive constant only depending on a, b, N_0 .

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To estimate the first derivatives of the solutions of system (1.20), we consider a linear system

$$\begin{cases} \frac{\partial S}{\partial t} + w(t, x) \frac{\partial S}{\partial x} = 0 \\ \frac{\partial Y}{\partial t} + z(t, x) \frac{\partial Y}{\partial x} = 0 \end{cases} \quad (2.32)$$

where z, w are regarded as given smooth functions. However, z, w are not arbitrary given. $S = z, Y = w$ itself is a solution of system(2.32). Assume that on the domain under consideration

$$w(t, x) - z(t, x) \geq \delta > 0 \quad (2.33)$$

where δ is positive constant. Then w and z are constant along characteristics respectively. Under these assumptions, system (2.32) enjoys the following remarkable properties:

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Lemma 2.3 Let

$$T_1 = \frac{\partial}{\partial t} + w(t, x) \frac{\partial}{\partial x}, T_2 = \frac{\partial}{\partial t} + z(t, x) \frac{\partial}{\partial x} \quad (2.34)$$

Then

$$[T_1, T_2] = T_1 T_2 - T_2 T_1 = 0 \quad (2.35)$$

For any Lipschitz continuous functions F and G , System (1.20) implies the conservation laws:

$$\begin{cases} \left(\frac{F(S)}{w-z} \right)_t + \left(\frac{wF(S)}{w-z} \right)_x = 0 \\ \left(\frac{G(Y)}{w-z} \right)_t + \left(\frac{zG(Y)}{w-z} \right)_x = 0 \end{cases} \quad (2.36)$$

For any fixed $T \geq 0$, we introduce

$$W_{\infty}(T) = \sup_{0 \leq t \leq T} \sup_{x \in R^+} \left\{ \left| \frac{\partial \lambda_+}{\partial x}(t, x) \right|, \left| \frac{\partial \lambda_-}{\partial x}(t, x) \right|, \left| \frac{\partial R_i}{\partial x}(t, x) \right|, \left| \frac{\partial S_i}{\partial x}(t, x) \right| \right\} \quad (2.39)$$

Lemma 2.4 Under the assumptions of Theorem 1.1, there exists a positive constant C only depending on N_0, a, b such that

$$W_{\infty}(T) \leq C(M + N) \quad (2.40)$$

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Proof: For any fixed point $(t, x) \in [0, T] \times R^+$, in the following we estimate $|\frac{\partial \lambda_-(t, x)}{\partial x}|$

Noting the third equation of system (1.20) and (2.35), (2.41), (2.42)

$$T_1 \lambda_-(t, x) = 0 \quad (2.49)$$

$$T_1(T_1 - T_2) \lambda_-(t, x) = 0 \quad (2.50)$$

$$(\frac{\partial}{\partial t} + \lambda_+(t, x) \frac{\partial}{\partial x})(\lambda_+(t, x) - \lambda_-(t, x)) \frac{\partial \lambda_-(t, x)}{\partial x} = 0 \quad (2.51)$$

$(\lambda_+(t, x) - \lambda_-(t, x)) \frac{\partial \lambda_-(t, x)}{\partial x}$ is a constant along any given forward characteristic.

There are only the following two cases:

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Case 1: The forward characteristic $\tilde{C}_1 : x = x_1(t, \beta)$ intersects x axis at a point $(0, \beta)$. By (2.51), we have

$$(\lambda_+(t, x) - \lambda_-(t, x)) \frac{\partial \lambda_-(t, x)}{\partial x} = (\Lambda_+(\beta) - \Lambda_-(\beta)) \Lambda'_-(\beta) \quad (2.52)$$

Noting Lemma 2.1, we have

$$\begin{aligned} & \left| \frac{\partial \lambda_-(t, x)}{\partial x} \right| \leq C |\Lambda'_-(\beta)| \\ & \leq C \left[\left(\sup_{\beta \in R^+} \left| \frac{\partial \Lambda_-(\beta)}{\partial f'} \right| \right) |f''(\beta)| + \left(\sup_{\beta \in R^+} \left| \frac{\partial \Lambda_-(\beta)}{\partial g} \right| \right) |g'(\beta)| \right] \end{aligned} \quad (2.53)$$

$$\leq C \sup_{\beta \in R^+} (|f''(\beta)| + |g'(\beta)|) \leq CN \quad (2.54)$$

where C only depends on N_0, a, b .

Case 2: The forward characteristic $\tilde{C}_1 : x = x_1(t)$ intersects t axis at a point $(\gamma, 0)$ and the backward characteristic $\tilde{C}_2 : x = x_2(t, \alpha)$ passing through $(\gamma, 0)$ intersects x axis at a point $(0, \alpha)$. Then, we have

$$(\lambda_+(t, x) - \lambda_-(t, x)) \frac{\partial \lambda_-(t, x)}{\partial x} = (\lambda_+(\gamma, 0) - \lambda_-(\gamma, 0)) \frac{\partial \lambda_-(\gamma, 0)}{\partial x} \quad (2.55)$$

$$\left| \frac{\partial \lambda_-(t, x)}{\partial x} \right| \leq C \left| \frac{\partial \lambda_-(\gamma, 0)}{\partial x} \right| \quad (2.56)$$

Noting the third equation of system (1.20), we can get

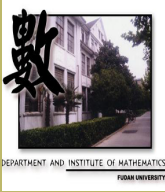
$$\left| \frac{\partial \lambda_-(t, x)}{\partial x} \right| \leq C \left| \frac{\partial \lambda_-(\gamma, 0)}{\partial \gamma} \right| \quad (2.57)$$

Noting (2.8), (2.9) and (2.12) we have

$$\left| \frac{\partial \lambda_-(t, 0)}{\partial t} \right| \leq C \left(\left| \frac{\partial}{\partial t} \left(\frac{2(h \cdot v)}{1 + |h|^2} \right) \right| + |\Lambda'_+(\alpha)| \left| \frac{d\alpha}{dt} \right| \right) \quad (2.58)$$

Noting $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\begin{cases} \frac{dx(t)}{dt} = \lambda_-(t, x_2(t, \alpha)) \\ t = 0 : x_2(0, \alpha) = \alpha \end{cases} \quad (2.59)$$


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Noting $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\begin{cases} \frac{dx(t)}{dt} = \lambda_-(t, x_2(t, \alpha)) \\ t = 0 : x_2(0, \alpha) = \alpha \end{cases} \quad (2.59)$$

Therefore

$$\left| \frac{\partial \lambda_-(t, 0)}{\partial t} \right| \leq C(|h'| |v| + |h| \left| \frac{\partial v}{\partial t} \right| + |\Lambda'_+(\alpha)|) \quad (2.60)$$

Noticing

$$v_i(\gamma, 0) = R_i(\gamma, 0) - \lambda_+(\gamma, 0)h_i(\gamma)$$

Then, by the estimate obtained in the previous case we can estimate $\left| \frac{\partial v}{\partial t} \right|$, thus

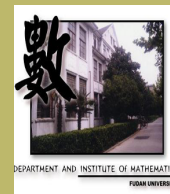
$$\left| \frac{\partial \lambda_-(t, 0)}{\partial t} \right| \leq C(|h'| + |h|N + |f''(\alpha)| + |g'(\alpha)|) \quad (2.61)$$

Therefore

$$\left| \frac{\partial \lambda_-(t, x)}{\partial x} \right| \leq C(M + N) \quad (2.62)$$

By the same method, we can get

$$\left| \frac{\partial S_i(t, x)}{\partial x} \right| \leq C(M + N) \quad (2.64)$$



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Proof of Theorem 1.1 Under the assumptions of Theorem 1.1, by Lemma 2.1-2.4, on the domain D

$$\|\lambda_{\pm}\|_1, \|R_i\|_1, \|S_i\|_1 \leq C(M + N + 1)$$

$$(\lambda_+(t, x) - \lambda_-(t, x)) \geq b + a$$

Noting (1.24), we can get uniform a priori estimate of C^1 norm of u and v . i.e. system (1.16) have the global C^1 solutions. Then, the system (1.5)-(1.7) have global C^2 solutions.

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★ **Uniform a priori estimate** For any fixed $T \geq 0$, we introduce

$$W_1(T) = \max_{i=1, \dots, n} \sup_{0 \leq t \leq T} \left\{ \int_0^{+\infty} \left| \frac{\partial \lambda_+}{\partial x}(t, x) \right| dx, \int_0^{+\infty} \left| \frac{\partial \lambda_-}{\partial x}(t, x) \right| dx \right. \\ \left. \int_0^{+\infty} \left| \frac{\partial R_i}{\partial x}(t, x) \right| dx, \int_0^{+\infty} \left| \frac{\partial S_i}{\partial x}(t, x) \right| dx \right\} \quad (3.1)$$

$$\tilde{W}_1(T) = \max_{i=1, \dots, n} \left\{ \sup_{\tilde{C}_1} \int_{\tilde{C}_1} \left| \frac{\partial \lambda_+}{\partial x}(t, x) \right| dt, \sup_{\tilde{C}_2} \int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x}(t, x) \right| dt \right. \\ \left. \sup_{\tilde{C}_1} \int_{\tilde{C}_1} \left| \frac{\partial R_i}{\partial x}(t, x) \right| dt, \sup_{\tilde{C}_2} \int_{\tilde{C}_2} \left| \frac{\partial S_i}{\partial x}(t, x) \right| dt \right\} \quad (3.2)$$

where \tilde{C}_1 stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ in the domain $[0, T] \times R^+$;

\tilde{C}_2 stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ in the domain $[0, T] \times R^+$.

Lemma 3.1 Under the assumptions of Theorem 1.3, there exists a positive constant C only depending on N_0, a, b such that, the following estimates hold:

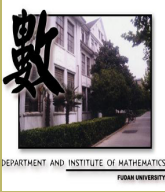
$$\tilde{W}_1(T), W_1(T) \leq C(N_2 + M_2 + M_1 N) \quad (3.3)$$

Proof: Differentiating the system (1.20) with respect to x . We have

$$\left\{ \begin{array}{l} \partial_t \left(\frac{\partial \lambda_-}{\partial x} \right) + \partial_x \left(\lambda_+ \frac{\partial \lambda_-}{\partial x} \right) = 0 \\ \partial_t \left(\frac{\partial \lambda_+}{\partial x} \right) + \partial_x \left(\lambda_- \frac{\partial \lambda_+}{\partial x} \right) = 0 \\ \partial_t \left(\frac{\partial R_i}{\partial x} \right) + \partial_x \left(\lambda_- \frac{\partial R_i}{\partial x} \right) = 0 \quad (i = 1, \dots, n) \\ \partial_t \left(\frac{\partial S_i}{\partial x} \right) + \partial_x \left(\lambda_+ \frac{\partial S_i}{\partial x} \right) = 0 \quad (i = 1, \dots, n) \end{array} \right. \quad (3.4)$$

We rewrite (3.5) as

$$\left\{ \begin{array}{l} d \left| \frac{\partial \lambda_-}{\partial x} \right| (dx - \lambda_+ dt) = 0 \\ d \left| \frac{\partial \lambda_+}{\partial x} \right| (dx - \lambda_- dt) = 0 \\ d \left| \frac{\partial R_i}{\partial x} \right| (dx - \lambda_- dt) = 0 \quad (i = 1, \dots, n) \\ d \left| \frac{\partial S_i}{\partial x} \right| (dx - \lambda_+ dt) = 0 \quad (i = 1, \dots, n) \end{array} \right. \quad (3.6)$$


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In the following we only prove

$$\int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt \leq C(M_2 + N_2 + M_1 N) \quad (3.16)$$

There are only three possibilities:

Case 1: For any fixed $\alpha \in R^+$, let $\tilde{C}_2: x = x_2(t, \alpha)$ stands for any given backward characteristic, passing through the point $A(0, \alpha)$ on the x axis and intersecting $t = T$ at a point P . We draw a forward characteristic $\tilde{C}_1: x = x_1(t, \beta)$ from P downward and intersects x axis at a point $B(0, \beta)$.

Then, we integrate Equation (3.15) in the region APB to get

$$\int_{\tilde{C}_2} (\lambda_+ - \lambda_-) \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt = \int_{\beta}^{\alpha} |\Lambda'_-(x)| dx \quad (3.17)$$

Notice that Lemma 2.1 and (2.2), (3.2), we can get

$$\int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt \leq CW_1(0) \leq CN_2 \quad (3.18)$$

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Case 2: For any fixed $\alpha \in R^+$, let $\tilde{C}_2: x = x_2(t, \alpha)$ stands for any given backward characteristic, passing through the point $A(0, \alpha)$ on the x axis and intersecting $t = T$ at a point P . We draw a backward characteristic $\tilde{C}_1: x = x_1(t)$ from P downward and intersects t axis at a point $B(\gamma, 0)$.

Then, we integrate Equation (3.15) in the region $PAOB$ to get

$$\int_{\tilde{C}_2} (\lambda_+ - \lambda_-) \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt = \int_0^\gamma \lambda_+ \left| \frac{\partial \lambda_-}{\partial x} \right| (t, 0) dt + \int_0^\alpha |\Lambda'_-(x)| dx \quad (3.19)$$

Using the third equation of system (1.20) and Lemma 2.1, we have

$$\int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt \leq C \left(\int_0^\gamma \left| \frac{\partial \lambda_-}{\partial t} \right| (t, 0) dt + \int_0^\alpha |\Lambda'_-(x)| dx \right) \quad (3.20)$$

Then, noting (2.61)

$$\begin{aligned} \int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt &\leq C \left(\int_0^{+\infty} \left| \frac{\partial \lambda_-}{\partial t} \right| (t, 0) dt + \int_0^{+\infty} |\Lambda'_-(x)| dx \right) \\ &\leq C(M_2 + N_2 + M_1 N) \end{aligned} \quad (3.21)$$

Case 3: For any fixed $\alpha \in R^+$, let \tilde{C}_2 : $x = x_2(t, \alpha)$ stands for any given backward characteristic, passing through the point $A(0, \alpha)$ on the x axis and intersecting t axis at a point $B(\gamma, 0)$.

Then, we integrate equation (3.15) in the region AOB to get

$$\int_0^\gamma (\lambda_+ - \lambda_-) \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt = \int_0^\alpha |\Lambda'_-(x)| dx + \int_0^\gamma \lambda_+ \left| \frac{\partial \lambda_-}{\partial x} \right| (t, 0) dt \quad (3.22)$$

Similarly, we can get

$$\int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt \leq C \left(\int_0^\alpha |\Lambda'_-(x)| dx + \int_0^\gamma \left| \frac{\partial \lambda_-}{\partial t} \right| (t, 0) dt \right)$$

Then

$$\int_{\tilde{C}_2} \left| \frac{\partial \lambda_-}{\partial x} \right| (t, x) dt \leq C(M_2 + N_2 + M_1 N) \quad (3.23)$$

Lemma 3.2 Under the assumptions of Theorem 1.3, we have

$$\left\{ \int_{L_1} (1 - \lambda_+)(t, x) dt, \int_{L_2} (1 + \lambda_-)(t, x) dt, \right. \\ \left. \int_{L_1} |R_i(t, x)| dt, \int_{L_2} |S_i(t, x)| dt \right\} \leq C(N_1 + M_1) \quad (3.31)$$

$$\left\{ \int_{\tilde{C}_1} (1 - \lambda_+)(t, x) dt, \int_{\tilde{C}_2} (1 + \lambda_-)(t, x) dt, \right. \\ \left. \int_{\tilde{C}_1} |R_i(t, x)| dt, \int_{\tilde{C}_2} |S_i(t, x)| dt \right\} \leq C(N_1 + M_1) \quad (3.32)$$

where \tilde{C}_1 stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ in the domain $[0, T] \times R^+$;

\tilde{C}_2 stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ in the domain $[0, T] \times R^+$; L_1 stands for any given radial that has the slope 1 in the domain $[0, T] \times R^+$; L_2 stands for any given radial that has the slope -1 in the domain $[0, T] \times R^+$.

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★ The proof of Theorem 1.3

Let

$$\frac{D}{D_1 t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \quad (4.1)$$

Obviously,

$$\frac{D}{D_1 t} = T_1 + (1 - \lambda_+) \frac{\partial}{\partial x} \quad (4.2)$$

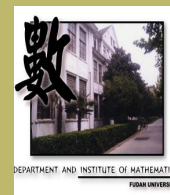
Thus, noting system (1.20)

$$\frac{D\lambda_-}{D_1 t} = T_1\lambda_- + (1 - \lambda_+) \frac{\partial\lambda_-}{\partial x} \quad (4.3)$$

In the following we consider Equation (4.3), i.e.

$$\frac{D\lambda_-}{D_1 t} = (1 - \lambda_+) \frac{\partial\lambda_-}{\partial x} \quad (4.4)$$

For any fixed point $(t, x) \in D$, define $\xi = x - t$

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Case 1: $\xi \geq 0$, it follows Equation (4.4) that

$$\lambda_-(t, x) = \lambda_-(t, \xi + t) = \lambda_-(0, \xi) + \int_0^t (1 - \lambda_+) \frac{\partial \lambda_-}{\partial x}(s, \xi + s) ds \quad (4.5)$$

By (2.65) and Lemma 3.2, we have

$$\begin{aligned} & \left| \int_0^t (1 - \lambda_+) \frac{\partial \lambda_-}{\partial x}(s, \xi + s) ds \right| \\ & \leq W_\infty(\infty) \int_0^{+\infty} |1 - \lambda_+|(s, \xi + s) dx \\ & \leq C(M + N)(M_1 + N_1) \end{aligned} \quad (4.6)$$

This implies that the integral $\int_0^t (1 - \lambda_+) \frac{\partial \lambda_-}{\partial x}(s, \xi + s) ds$ converges uniformly for $\xi \in R^+$. On the other hand, noting that all functions in the right-hand side in Equation (4.5) are continuous with respect to ξ . Then, we observe that there exists a unique function $\tilde{\psi}(\xi) \in C^0(R^+)$ such that

$$\lambda_-(t, x) \longrightarrow \tilde{\psi}(x - t) \quad t \longrightarrow \infty \quad (4.7)$$

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Case 2: $\xi \leq 0$, it follows Equation (4.5) that

$$\lambda_-(t, x) = \lambda_-(t, \xi + t) = \lambda_-(-\xi, 0) + \int_{-\xi}^t (1 - \lambda_+) \frac{\partial \lambda_-}{\partial x}(s, \xi + s) ds \quad (4.8)$$

By (2.65) and Lemma 3.2, we can get

$$\begin{aligned} & \int_{-\xi}^t (1 - \lambda_+) \frac{\partial \lambda_-}{\partial x}(s, \xi + s) ds \\ & \leq W_\infty(\infty) \int_{-\xi}^t (1 - \lambda_+)(s, \xi + s) ds \\ & \leq C(M + N)(M_1 + N_1) \end{aligned} \quad (4.9)$$

Then, we obtain that there exists a unique function $\bar{\psi}(\xi) \in C^0(R^-)$ such that

$$\lambda_-(t, x) \longrightarrow \bar{\psi}(x - t) \quad t \longrightarrow +\infty \quad (4.10)$$

Case 3: When $\xi \longrightarrow 0$, notice that above cases we can get

$$\tilde{\psi}(\xi) \longrightarrow \tilde{\psi}(0) \quad \text{and} \quad \bar{\psi}(\xi) \longrightarrow \bar{\psi}(0) \quad (4.11)$$

Moreover

$$\tilde{\psi}(0) = \bar{\psi}(0) \quad (4.12)$$

We define

$$\psi(\xi) = \begin{cases} \tilde{\psi}(\xi), & \xi \in R^+; \\ \bar{\psi}(-\xi), & \xi \in R^-; \end{cases}$$

Hence from above we have proved the following lemma

Lemma 4.1 There exists a unique function $\psi(x - t) \in C^0(R)$, such that

$$\lambda_-(t, x) \longrightarrow \psi(x - t) \quad t \longrightarrow +\infty \quad (4.13)$$

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Remark 4.1 In the same way, we can obtain that there exists a unique function $\psi_i(x - t) \in C^0(R)$ such that

$$S_i(t, x) \longrightarrow \psi_i(x - t) \quad t \longrightarrow +\infty \quad i = 1, \dots, n \quad (4.14)$$

Lemma 4.2 When $t \longrightarrow +\infty$, we have

$$\lambda_+(t, x) \longrightarrow 1 \quad (4.15)$$

$$R_i(t, x) \longrightarrow 0 \quad i = 1, \dots, n \quad (4.16)$$

uniformly for all $x \geq 0$.

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Noting (1.24),

$$\lim_{t \rightarrow \infty} u_i(t, x) = \lim_{t \rightarrow \infty} \frac{R_i(t, x) - S_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)} = \lim_{t \rightarrow \infty} \frac{-\psi_i(x - t)}{1 - \psi(x - t)} \quad (4.17)$$

$$\begin{aligned} \lim_{t \rightarrow \infty} v_i(t, x) &= \lim_{t \rightarrow \infty} \frac{\lambda_+ S_i(t, x) - \lambda_- R_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)} \\ &= \lim_{t \rightarrow \infty} \frac{\psi_i(x - t)}{1 - \psi(x - t)} \end{aligned} \quad (4.18)$$

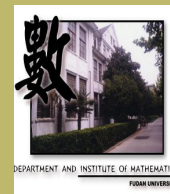
Then, when $x \geq 0$, we can get

$$\lim_{t \rightarrow \infty} u_i(t, x) = \frac{-\psi_i(x - t)}{1 - \psi(x - t)} \doteq \Phi_i(x - t) \quad i = 1, \dots, n \quad (4.19)$$

$$\lim_{t \rightarrow \infty} v_i(t, x) = \frac{\psi_i(x - t)}{1 - \psi(x - t)} \doteq -\Phi_i(x - t) \quad i = 1, \dots, n \quad (4.20)$$

We next prove that $\Phi_i(\xi) \in C^1(R)$. Noting $\psi_i(\xi), \psi(\xi) \in C^0(R)$, we need to show that $d(\psi_i(\xi))/d\xi, d(\psi(\xi))/d\xi \in C^0(R)$. It suffices to show that $\psi(\xi), \psi_i(\xi) \in C^1(R)$.

In the following we only prove $\psi(\xi) \in C^1(R)$


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Lemma 4.3 Under the assumptions of Theorem 1.3, the limit

$$\lim_{t \rightarrow \infty} \frac{\partial \lambda_-}{\partial x}(t, x_1(t, \beta)) \doteq \psi^*(\beta) \quad (4.21)$$

exists and is continuous. Moreover

$$|\psi^*(\beta)| \leq C(M + N)(N_2 + M_2 + M_1 N) \quad (4.22)$$

Lemma 4.4 The limit

$$\lim_{t \rightarrow \infty} \frac{\partial \lambda_-}{\partial x}(t, \xi + t)$$

exists and is continuous with respect to ξ .

Lemma 4.5

$$\frac{d\psi(\xi)}{d\xi} = \lim_{t \rightarrow \infty} \frac{\partial \lambda_-}{\partial x}(t, \xi + t). \quad (4.65)$$

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Remark 4.3 In the similar way, we also can prove

$$\lim_{t \rightarrow \infty} \frac{\partial S_i}{\partial x}(t, \xi + t) = \frac{d\psi_i(\xi)}{d\xi} \quad (4.69)$$

Lemma 4.6

$$\lim_{t \rightarrow \infty} \frac{\partial \lambda_-}{\partial x}(t, \xi + t) = \psi^*(\vartheta(\xi)) \quad (4.70)$$

is continuous in R . Moreover

$$\frac{d\psi(\xi)}{d\xi} = \psi^*(\vartheta(\xi)) \quad (4.71)$$

Remark 4.4 By the same method, we obtain that $\frac{\partial S_i}{\partial x}(t, \xi + t)$ have the similar conclusion. Moreover

$$\frac{d\psi_i(\xi)}{d\xi} = \psi_i^*(\vartheta(\xi)) \quad (4.75)$$

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