Initial-boundary value problem for the equation of time-like extremal surfaces in Minkowski space

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We denote $(x_0, x_1, ..., x_{n+1})$ a point in the $(n+1)+1$ dimensional Minkowski space endorsed with the metric

$$ds^2 = -dx_0^2 + dx_1^2 + ... + dx_{n+1}^2$$

Let

$$x_0 = t, x_1 = x, x_2 = \phi_1(t, x), ..., x_{n+1} = \phi_n(t, x)$$

be a two dimensional time like surface. Then the induced metric on the surface is

$$d_* s^2 = -dt^2 + dx^2 + d(\phi_1)^2 + ... + d(\phi_n)^2$$
$$= -(1 - (\phi_t)^2)dt^2 + (1 + (\phi_x)^2)dx^2 + 2\phi_t \cdot \phi_x dxdt$$

where $\phi = (\phi_1, ..., \phi_n)^T$, ϕ_t or ϕ_x denote partial dedifferentiation with respect to t or x respectively and $\cdot$ denotes inner product in R^n. Thus, it is easy to see that the area of the surface is

$$\int\int \sqrt{1 - (\phi_t)^2 + (\phi_x)^2 - (\phi_t)^2(\phi_x)^2 + (\phi_t \cdot \phi_x)^2} dxdt$$

An extremal surface is defined to be the extremal point of the area functional, hence it satisfies the Euler-Lagrange equations

$$\left(\frac{(1+(\phi_x)^2)\phi_t-(\phi_t\cdot\phi_x)\phi_x}{\sqrt{1-(\phi_t)^2+(\phi_x)^2-(\phi_t)^2(\phi_x)^2+(\phi_t\cdot\phi_x)^2}}\right)_t$$

$$-\left(\frac{(1-(\phi_t)^2)\phi_x+(\phi_t\cdot\phi_x)\phi_t}{\sqrt{1-(\phi_t)^2+(\phi_x)^2-(\phi_t)^2(\phi_x)^2+(\phi_t\cdot\phi_x)^2}}\right)_x=0 \qquad (4)$$

We consider the Cauchy problem for system (4) with initial data

$$\phi(0,x)=h(x),\ \ \phi_t(0,x)=g(x)$$

where h' and g are vector valued C^1 functions.

Let

$$u = \phi_x \ , \ v = \phi_t$$

Then, Eq.(4) can be equivalently rewritten as a first order systems of conservation laws for the unknown $U(t,x) = (u(t,x), v(t,x))$ as follows

$$u_t - v_x = 0$$

$$\left(\frac{(1+u^2)v - (u \cdot v)u}{\sqrt{1 - v^2 + u^2 - v^2u^2 + (u \cdot v)^2}}\right)_t - \left(\frac{(1-v^2)u + (u \cdot v)v}{\sqrt{1 - v^2 + u^2 - v^2u^2 + (u \cdot v)^2}}\right)_x = 0$$

The initial condition then becomes $(u(0,x), v(0,x)) = U_0(x) = (h'(x), g(x))$. This is an interesting model in Lorentian geometry.

♦ Barbashov, Nesterenko and Chervyakov [10](Commun.Math.Phys.(1982))
Milnor [11] (Michigan Math. (1990))

♦ Gu [12] (Nonlinear Differential Equations Appl 4. (1990)). [13] (Chinese Ann.Math (1994))
Kong [14] (Europhys.Lett. (2004))

♦ D. Chae and H. Huh [15] (J. Math. Phys(2004))
H.Lindblad [16] (Proc. Am. Math. Soc (2004))
Kong, Sun and Zhou [17] (J.Math.phys (2006))

In [17] they showed that system can be reduced to

$$\begin{cases} u_t - v_x = 0 \\ v_t - \dfrac{2(u \cdot v)}{1+u^2} v_x - \dfrac{1-v^2}{1+u^2} u_x = 0 \end{cases}$$

They found that it enjoys many interesting properties:nonstrictly hyperbolicity, constant multiplicity of eigenvalues, linear degeneracy of all characteristic fields, richness, etc. The system have two n-constant multiple eigenvalues:

$$\lambda_\pm = \frac{1}{1+u^2}(-(u \cdot v) \pm \sqrt{\triangle(u,v)})$$

where $\triangle(u,v) = 1 - v^2 + u^2 - u^2v^2 + (u \cdot v)^2 > 0$. They also proved

$$|\lambda_\pm(t,x)| \le 1$$

Let

$$R_i = v_i + \lambda_+ u_i, \quad (i = 1, ..., n)$$
$$S_i = v_i + \lambda_- u_i, \quad (i = 1, ..., n)$$

then they satisfies the following systems

$$\begin{cases} \partial_t \lambda_+ + \lambda_- \partial_x \lambda_+ = 0 \\ \partial_t R_i + \lambda_- \partial_x R_i = 0 \quad (i = 1, ..., n) \\ \partial_t \lambda_- + \lambda_+ \partial_x \lambda_- = 0 \\ \partial_t S_i + \lambda_+ \partial_x S_i = 0 \quad (i = 1, ..., n) \end{cases}$$

$$t = 0 : \lambda_+(0, x) = \Lambda_+(x), \lambda_-(0, x) = \Lambda_-(x),$$
$$R_i(0, x) = R_i^0(x), \;\; S_i(0, x) = S_i^0(x)$$

Then it was proved that above system admits a global classical solution for all $t \in R^+$, provided that U_0 is C^1 and the strictly hyperbolic condition

$$\delta = \inf_{x \in R} \Lambda_+(x) - \sup_{x \in R} \Lambda_-(x) > 0 \tag{5}$$

is satisfied.

$$u_i(t, x) = \frac{R_i(t, x) - S_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)}, \;\; v_i(t, x) = \frac{\lambda_+ S_i(t, x) - \lambda_- R_i(t, x)}{\lambda_+(t, x) - \lambda_-(t, x)}$$

Theorem A. Suppose that (5) is satisfied, then the cauchy problem (4) admits a unique global C^2 solution $\phi = \phi(t,x)$ on $R^+ \times R$. Moreover, it holds that

$$\triangle(\phi_x(t,x), \phi_t(t,x)) > 0, \ \ \forall (t,x) \in R^+ \times R$$

Under the following assumptions:

$$\sup_{x \in R}\{|h''(x)| + |g'(x)|\} \doteq \bar{M} < \infty,$$

$$\int_{-\infty}^{+\infty} |h'(x)| + |g(x)| dx \doteq \bar{N}_1 < \infty$$

$$\int_{-\infty}^{+\infty} |h''(x)| + |g'(x)| dx \doteq \bar{N}_2 < \infty$$

$$\bar{M}_0 = \sup_{x \in R}\{|h'(x)| + |g(x)|\}$$

Firstly we consider the Cauchy problem:

- Consider Cauchy problem of the first order general quasilinear hyperbolic systems

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = B(u),$$
$$u(0, x) = f(x)$$

♦The existence of the global classical solutions:
Bressan [1]Indiana University Mathamatics Journal (1988)
Li [2] (published in the United States with John Wiley & Sons, 1994.)
Li, Zhou and Kong [3] Comm.PDE (1994). [4]Nonl.Anal.(1997)
Kong [5] ,
Zhou [6] Chin. Ann. Math. (2004)

♦ Asymptotic behavior:
Kong and Yang [7] Comm in Part Diff Eqs. (2003)
Dai and Kong [8] Chin. Ann. Math. B (2006), [9] (preprint).

★★In the following we consider the following diagonalizable quasilinear hyperbolic systems

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = 0 \tag{1}$$

where $u = (u_1, ..., u_n)^T$ is the unknown vector-valued function of (t, x). $\lambda_i(u)$ is given C^2 vector-valued function of u and is linearly degenerate, i.e.

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0$$

and the system (1) is strictly hyperbolic, i.e.

$$\lambda_1(u) < ... < \lambda_n(u)$$

Suppose that there exists a positive constant δ such that,

$$\lambda_{i+1}(u) - \lambda_i(v) \geq \delta, \quad i = 1, ..., n-1.$$

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Consider the cauchy problem for the system (1) with the following initial data

$$t = 0 : \quad u = f(x)$$

where $f(x)$ is a C^1 vector-valued function of x. The global existence of the classical solutions is well-known see Li [2].

Assumptions

$$\sup_{x\in R} |f'(x)| \doteq M < \infty$$

$$\int_{-\infty}^{+\infty} |f(x)|dx \doteq N_1 < \infty$$

$$\int_{-\infty}^{+\infty} |f'(x)|dx \doteq N_2 < \infty$$

$$\sup_{x\in R} |f(x)| \doteq M_0$$

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Theorem 1.1. Under the assumptions of above, there exists a unique C^1 vector-valued function $\phi(x) = (\phi_1(x), ..., \phi_n(x))^T$ such that,

$$u(t,x) \longrightarrow \sum_{i=1}^{n} \phi_i(x - \lambda_i(0)t)e_i, \quad t \longrightarrow \infty$$

where

$$e_i = (0, ..., 1^i, 0, ..., 0)^T$$

Moreover, $\phi_i(x)(i = 1, ..., n)$ is global Lipschitz continuous. Furthermore, If system (1) is rich and the derivative of the initial data $f'(x)$ isglobal ρ-hölder continuous, where $0 < \rho \leq 1$, i.e. there exists a positive constant κ independent of $\alpha, \beta \in R$ such that,

$$|f'(\alpha) - f'(\beta)| \leq \kappa|\alpha - \beta|^\rho$$

Then $\phi_i'(x)$ $(i = 1, ..., n)$ satisfies

$$|\phi_i'(\alpha) - \phi_i'(\beta)| \leq C\tilde{M}(|\alpha - \beta|^\rho + |\alpha - \beta|)$$

Lemma 4.1. Under the assumptions above, the limit

$$\lambda_-(t,x) \longrightarrow \tilde{\psi}(x-t), \quad t \longrightarrow \infty$$

exists and for any $\alpha, \beta \in R$, we have

$$|\tilde{\psi}(\alpha) - \tilde{\psi}(\beta)| \le C\bar{M}|\alpha - \beta|$$

Moreover,

$$|\tilde{\psi}(\alpha)| \le 1$$

Remark 4.1. Using the similar method, we can get the following estimate

$$S_i(t,x) \longrightarrow \tilde{\phi}_i(x-t),\ \text{exist and } |\tilde{\phi}_i(\alpha) - \tilde{\phi}_i(\beta)| \le C\bar{M}|\alpha - \beta|$$

$$\lambda_+(t,x) \longrightarrow \psi(x+t),\ \text{exist and } |\psi(\alpha) - \psi(\beta)| \le C\bar{M}|\alpha - \beta|$$

$$R_i(t,x) \longrightarrow \check{\phi}_i(x+t),\ \text{exist and } |\check{\phi}_i(\alpha) - \check{\phi}_i(\beta)| \le C\bar{M}|\alpha - \beta|$$

Moreover,

$$|\psi(\alpha)| \le 1,\ |\tilde{\phi}_i(\alpha)|, |\check{\phi}_i(\alpha)| \le 1 + C\bar{M}\bar{N}_1$$

Then

$$\lim_{t\longrightarrow\infty} u_i(t,x) = \lim_{t\longrightarrow\infty} \frac{\check{\phi}_i(x+t) - \tilde{\phi}_i(x-t)}{\psi(x+t) - \tilde{\psi}(x-t)}$$

$$\lim_{t\longrightarrow\infty} v_i(t,x) = \lim_{t\longrightarrow\infty} \frac{\psi(x+t)\tilde{\phi}_i(x-t) - \tilde{\psi}(x-t)\check{\phi}_i(x+t)}{\psi(x+t) - \tilde{\psi}(x-t)}$$

Then, we can get

when $x \geq 0$

$$\lim_{t\longrightarrow\infty} u_i(t,x) = \frac{-\tilde{\phi}_i(x-t)}{1-\tilde{\psi}(x-t)} \doteq \Phi_i(x-t), \quad i = 1, ..., n$$

$$\lim_{t\longrightarrow\infty} v_i(t,x) = \frac{\tilde{\phi}_i(x-t)}{1-\tilde{\psi}(x-t)} \doteq -\Phi_i(x-t), \quad i = 1, ..., n$$

when $x \leq 0$

$$\lim_{t\longrightarrow\infty} u_i(t,x) = \frac{\check{\phi}_i(x+t)}{1+\psi(x+t)} \doteq \Psi_i(x+t), \quad i = 1, ..., n$$

$$\lim_{t\longrightarrow\infty} v_i(t,x) = \frac{\check{\phi}_i(x+t)}{1+\psi(x+t)} \doteq \Psi_i(x+t), \quad i = 1, ..., n$$

So we can conclude

$$u_i(t,x) \longrightarrow \Phi_i(x-t) + \Psi_i(x+t)$$

$$v_i(t,x) \longrightarrow -\Phi_i(x-t) + \Psi_i(x+t)$$

We also have conclusion that $\Phi_i(\alpha)$ and $\Psi_i(\alpha) (i = 1, ..., n)$ are Lipschitz continuous.

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Theorem 1.2. Under the assumptions of above, furthermore, suppose initial data satisfies (5). There exist unique C^1 vector-valued functions $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))$ and $\Psi(x) = (\Psi_1(x), ..., \Psi_n(x))$ such that,

$$(\phi_i)_x \longrightarrow \Phi_i(x-t) + \Psi_i(x+t), \quad t \longrightarrow \infty \quad i = 1, ..., n$$

$$(\phi_i)_t \longrightarrow -\Phi_i(x-t) + \Psi_i(x+t) \;\; t \longrightarrow \infty \;\; i = 1, ..., n$$

Moreover $\Phi_i(x)$ and $\Psi_i(x)(i = 1, ..., 2n)$ are global Lipschitz continuous. Exactly, there exists a positive constant $\tilde{M}$ only depending on $\bar{M}, \bar{N}_1, \bar{N}_2$. It holds that

$$|\Phi_i(\alpha) - \Phi_i(\beta)| \le C\tilde{M}|\alpha - \beta|$$
$$|\Psi_i(\alpha) - \Psi_i(\beta)| \le C\tilde{M}|\alpha - \beta|$$

Furthermore, If the initial data h'', g' are global ρ-hölder $(0 < \rho \le 1)$ continuous, i.e. there exists a positive constant κ independent of $\alpha, \beta \in R$ such that,

$$|h''(\alpha) - h''(\beta)| + |g'(\alpha) - g'(\beta)| \le \kappa|\alpha - \beta|^\rho$$

then, $\Phi_i'(x)$ and $\Psi_i'(x)$ satisfy

$$|\Phi_i'(\alpha) - \Phi_i'(\beta)| + |\Psi_i'(\alpha) - \Psi_i'(\beta)| \le C\kappa\tilde{M}^\rho|\alpha - \beta|^\rho + C\tilde{M}|\alpha - \beta|$$

★★**Initial-boundary value problem for the equation of time-like extremal surface in Minkowski space**

We only consider the initial-boundary value problem with Neumann boundary condition

$$\begin{cases} u_t - v_x = 0 \\ v_t - \dfrac{2(u\cdot v)}{1+u^2}v_x - \dfrac{1-v^2}{1+u^2}u_x = 0 \\ t=0: \quad u = f'(x), v = g(x) \\ x = 0: \quad u = h(t) \end{cases} \tag{1.16}$$

Suppose that U_0, h are C^1 functions with bounded C^1 norm and the initial conditions satisfy

$$\sup_{x\in R^+} \Lambda_-(x) \leq -a < 0 < b \leq \inf_{x\in R^+} \Lambda_+(x) \tag{1.25}$$

Without loss of generality, we assume $a < b$. (Otherwise, we can always replace a by a smaller positive number). If the Neumann boundary data is sufficiently small, for example

$$|h(t)| \leq \frac{b-a}{3} \tag{1.26}$$

Then we have the following global existence result for the initial-boundary value problem (1.5)-(1.7)

Theorem 1.1 Suppose that the initial data and Neumann boundary data satisfy (1.25), (1.26), the conditions of C^2 compatibility (1.9) are satisfied, then the initial-boundary value problem (1.5)-(1.7) admits a unique global C^2 solution $\phi = \phi(t, x)$ on $R^+ \times R^+$.

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Similarly, we suppose U_0, H' are C^1 functions with bounded C^1 norm and the initial conditions satisfy

$$\sup_{x\in R^+} \Lambda_-(x) \leq -a < 0 < b \leq \inf_{x\in R^+} \Lambda_+(x)$$

Without loss of generality, we assume $a < b$. If the first derivative of Dirichlet boundary data is sufficiently small, for example

$$|H'(t)| \leq b - a \tag{1.27}$$

The conditions of C^2 compatibility are satisfied. i.e.

$$f(0) = H(0), \quad H'(0) = g(0) \tag{1.28}$$

and

$$H''(0) - \frac{2f'(0)\cdot g(0)}{1+f'^2(0)}g'(0) - \frac{1-g^2(0)}{1+f'^2(0)}f''(0) = 0 \tag{1.29}$$

Then we have the following global existence result

Theorem 1.2 Suppose the above assumptions (1.25) and (1.27)-(1.29) are satisfied, then the initial-boundary value problem (1.5), (1.6) and (1.8) admits a unique global C^2 solution $\phi = \phi(t, x)$ on $R^+ \times R^+$.

If we also suppose that the initial and boundary datum satisfy the following assumptions:

$$\sup_{x\in R^+}\{|f''(x)|+|g'(x)|\} \doteq N < \infty \tag{1.30}$$

$$\int_0^{+\infty} |f'(x)|+|g(x)|dx \doteq N_1 < \infty \tag{1.31}$$

$$\int_0^{+\infty} |f''(x)|+|g'(x)|dx \doteq N_2 < \infty \tag{1.32}$$

$$\sup_{x\in R^+}\{|f'(x)|+|g(x)|\} = N_0 \tag{1.33}$$

$$\sup_{t\in R^+}\{|h'(t)|\} \doteq M < \infty \tag{1.34}$$

$$\int_0^{+\infty} |h(t)|dt \doteq M_1 < \infty \tag{1.35}$$

$$\int_0^{+\infty} |h'(t)|dt \doteq M_2 < \infty \tag{1.36}$$

Based on the existence of the global classical solutions, we also prove the following Theorem:

Theorem 1.3 Under the assumptions of Theorem 1.1 and above, there exists a unique C^1 vector-valued function $\Phi(x) = (\Phi_1(x), ..., \Phi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Phi_i(x-t), -\Phi_i(x-t)) \quad i = 1, ..., n \tag{1.37}$$

Similarly, under the assumptions that

$$\sup_{x\in R^+}\{|f''(x)|+|g'(x)|\} \doteq N < \infty$$

$$\int_0^{+\infty} |f'(x)|+|g(x)|dx \doteq N_1 < \infty$$

$$\int_0^{+\infty} |f''(x)|+|g'(x)|dx \doteq N_2 < \infty$$

$$\sup_{x\in R^+}\{|f'(x)|+|g(x)|\} = N_0$$

$$\sup_{t\in R^+}\{|H''(t)|\} \doteq M < \infty \tag{1.38}$$

$$\int_0^{+\infty} |H'(t)|dt \doteq M_1 < \infty \tag{1.39}$$

$$\int_0^{+\infty} |H''(t)|dt \doteq M_2 < \infty \tag{1.40}$$

We have the following Theorem

Theorem 1.4 Under the assumptions of Theorem 1.2 and above, there exists a unique C^1 vector-valued function $\Psi(x) = (\Psi_1(x), ..., \Psi_n(x))$ such that

$$((\phi_i)_x, (\phi_i)_t) \longrightarrow (\Psi_i(x-t), -\Psi_i(x-t)) \quad i = 1, ..., n \tag{1.41}$$

★**Key idea of the proof**

Lemma 2.1 Under the assumptions of (1.9), (1.25) and (1.26), system (1.20) is strictly hyperbolicity. Furthermore, on the domain D we have

$$\lambda_-(t,x) \leq -a < 0 < b \leq \lambda_+(t,x) \tag{2.3}$$

Proof: For any fixed $(t, x) \in D$, we draw the forward characteristic $\tilde{C}_1 : x = x_1(t)$ through this point. There are only two possibilities:

Case 1: The forward characteristic $\tilde{C}_1 : x = x_1(t, \beta)$ intersects x axis at a point $(0, \beta)$. By the second equation of system (2.1), $\lambda_-(t, x)$ is a constant along any given forward characteristic, then we have

$$\lambda_-(t, x) = \Lambda_-(\beta) \tag{2.4}$$

Noting (1.25)

$$\lambda_-(t, x) \leq -a \quad \forall\, (t, x) \in D \tag{2.5}$$

In the similar way, we draw the backward characteristic $\tilde{C}_2 : x = x_2(t, \alpha)$ intersects x axis at a point $(0, \alpha)$. Along any given backward characteristic $\lambda_+(t, x)$ is a constant

$$\lambda_+(t, x) = \Lambda_+(\alpha) \tag{2.6}$$

Then

$$b \leq \lambda_+(t, x) \quad \forall\, (t, x) \in D \tag{2.7}$$

Case 2: The forward characteristic $\tilde{C}_1 : x = x_1(t)$ intersects t axis at a point $(\gamma, 0)$ and the backward characteristic $\tilde{C}_2 : x = x_2(t)$ passing through $(\gamma, 0)$ intersects x axis at a point $(0, \alpha)$, where $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\frac{dx_2(t)}{dt} = \lambda_-(t, x_2(t, \alpha)), \quad x_2(0, \alpha) = \alpha$$

Similarly, we can get

$$\lambda_-(t, x) = \lambda_-(\gamma, 0) \tag{2.8}$$

$$\lambda_+(\gamma, 0) = \Lambda_+(\alpha) \tag{2.9}$$

Then

$$\lambda_-(t, x) + \Lambda_+(\alpha) = \frac{-2(h \cdot v)}{1 + |h|^2} \tag{2.12}$$

Let $\theta \in [0, \pi]$ is the angle between vectors h and v. We have $h \cdot v = |h||v| \cos\theta$. Then, Equation (2.10) can be rewritten as

$$(1 + |h|^2)\Lambda_+(\alpha) + |h||v| \cos\theta = \sqrt{1 - |v|^2 + |h|^2 - |v|^2|h|^2 + |h|^2|v|^2 \cos^2\theta} \tag{2.13}$$

Then

$$(1+|h|^2)^2\Lambda_+^2(\alpha)+(1+|h|^2)|v|^2+2(1+|h|^2)\Lambda_+(\alpha)|h||v| \cos\theta = 1+|h|^2 \tag{2.14}$$

Dividing (2.14) by $1 + |h|^2$ leads to

$$(1 + |h|^2)\Lambda_+^2(\alpha) + |v|^2 + 2\Lambda_+(\alpha)|h||v| \cos\theta = 1 \tag{2.15}$$

Noting $\theta \in [0, \pi]$, then $|\cos\theta| \leq 1$

$$(1 + |h|^2)\Lambda_+^2(\alpha) + |v|^2 - 2\Lambda_+(\alpha)|h||v| \leq 1$$

$$(|v| - |h|\Lambda_+(\alpha))^2 \leq 1 - \Lambda_+^2(\alpha)$$

Notice that the initial data satisfies (1.25), we can get

$$|v| \leq \Lambda_+(\alpha)|h| + \sqrt{1 - \Lambda_+^2(\alpha)} \tag{2.16}$$

Thus

$$|v| \leq |h| + 1 \tag{2.17}$$

By Equation (2.12), we have

$$-\lambda_-(t,x) = \Lambda_+(\alpha) + \frac{2|h||v|\cos\theta}{1+|h|^2} \tag{2.18}$$

$$\geq \Lambda_+(\alpha) - \frac{2|h||v|}{1+|h|^2}$$

Noting (1.25) and (2.17)

$$-\lambda_-(t,x) \geq b - \frac{2|h|(|h|+1)}{1+|h|^2} \tag{2.19}$$

On the other hand, the Neumann boundary data satisfies (1.26), then

$$\frac{2|h|(|h|+1)}{1+|h|^2} \leq b - a$$

Therefore

$$\lambda_-(t,x) \leq -a \tag{2.20}$$

Lemma 2.2 Let R_i, S_i be as system (1.20), then

$$\{|R_i(t,x)|, |S_i(t,x)|\} \leq C \tag{2.21}$$

where C is a positive constant only depending on a, b, N_0.

To estimate the first derivatives of the solutions of system (1.20), we consider a linear system

$$\begin{cases} \frac{\partial S}{\partial t} + w(t,x)\frac{\partial S}{\partial x} = 0 \\ \frac{\partial Y}{\partial t} + z(t,x)\frac{\partial Y}{\partial t} = 0 \end{cases} \tag{2.32}$$

where z, w are regarded as given smooth functions. However, z, w are not arbitrary given. $S = z$, $Y = w$ itself is a solution of system(2.32). Assume that on the domain under consideration

$$w(t,x) - z(t,x) \geq \delta > 0 \tag{2.33}$$

where δ is positive constant. Then w and z are constant along characteristics respectively. Under these assumptions, system (2.32) enjoys the following remarkable properties:

Lemma 2.3 Let

$$T_1 = \frac{\partial}{\partial t} + w(t,x)\frac{\partial}{\partial x}, T_2 = \frac{\partial}{\partial t} + z(t,x)\frac{\partial}{\partial x} \tag{2.34}$$

Then

$$[T_1, T_2] = T_1T_2 - T_2T_1 = 0 \tag{2.35}$$

For any Lipschitz continuous functions F and G, System (1.20) implies the conservation laws:

$$\begin{cases} (\dfrac{F(S)}{w-z})_t + (\dfrac{wF(S)}{w-z})_x = 0 \\ (\dfrac{G(Y)}{w-z})_t + (\dfrac{zG(Y)}{w-z})_x = 0 \end{cases} \tag{2.36}$$

For any fixed $T \geq 0$, we introduce

$$W_\infty(T) = \sup_{0 \leq t \leq T} \sup_{x \in R^+} \{|\frac{\partial \lambda_+}{\partial x}(t,x)|, |\frac{\partial \lambda_-}{\partial x}(t,x)|, |\frac{\partial R_i}{\partial x}(t,x)|, |\frac{\partial S_i}{\partial x}(t,x)|\} \tag{2.39}$$

Lemma 2.4 Under the assumptions of Theorem 1.1, there exists a positive constant C only depending on N_0, a, b such that

$$W_\infty(T) \leq C(M+N) \tag{2.40}$$

Proof: For any fixed point $(t,x) \in [0,T] \times R^+$, in the following we estimate $|\frac{\partial \lambda_-(t,x)}{\partial x}|$

Noting the third equation of system (1.20) and (2.35), (2.41), (2.42)

$$T_1 \lambda_-(t,x) = 0 \tag{2.49}$$

$$T_1(T_1 - T_2)\lambda_-(t,x) = 0 \tag{2.50}$$

$$(\frac{\partial}{\partial t} + \lambda_+(t,x)\frac{\partial}{\partial x})(\lambda_+(t,x) - \lambda_-(t,x))\frac{\partial \lambda_-(t,x)}{\partial x} = 0 \tag{2.51}$$

$(\lambda_+(t,x) - \lambda_-(t,x))\frac{\partial \lambda_-(t,x)}{\partial x}$ is a constant along any given forward characteristic. There are only the following two cases:

Case 1: The forward characteristic $\tilde{C}_1 : x = x_1(t, \beta)$ intersects x axis at a point $(0, \beta)$. By (2.51), we have

$$(\lambda_+(t,x) - \lambda_-(t,x))\frac{\partial \lambda_-(t,x)}{\partial x} = (\Lambda_+(\beta) - \Lambda_-(\beta))\Lambda'_-(\beta) \tag{2.52}$$

Noting Lemma 2.1, we have

$$|\frac{\partial \lambda_-(t,x)}{\partial x}| \le C|\Lambda'_-(\beta)|$$

$$\le C[(\sup_{\beta \in R^+} |\frac{\partial \Lambda_-(\beta)}{\partial f'}|)|f''(\beta)| + (\sup_{\beta \in R^+} |\frac{\partial \Lambda_-(\beta)}{\partial g}|)|g'(\beta)|] \tag{2.53}$$

$$\le C \sup_{\beta \in R^+} (|f''(\beta)| + |g'(\beta)|) \le CN \tag{2.54}$$

where C only depends on N_0, a, b.

Case 2: The forward characteristic $\tilde{C}_1 : x = x_1(t)$ intersects t axis at a point $(\gamma, 0)$ and the backward characteristic $\tilde{C}_2 : x = x_2(t, \alpha)$ passing through $(\gamma, 0)$ intersects x axis at a point $(0, \alpha)$. Then, we have

$$(\lambda_+(t,x) - \lambda_-(t,x))\frac{\partial \lambda_-(t,x)}{\partial x} = (\lambda_+(\gamma,0) - \lambda_-(\gamma,0))\frac{\partial \lambda_-(\gamma,0)}{\partial x} \tag{2.55}$$

$$|\frac{\partial \lambda_-(t,x)}{\partial x}| \leq C|\frac{\partial \lambda_-(\gamma,0)}{\partial x}| \tag{2.56}$$

Noting the third equation of system (1.20), we can get

$$|\frac{\partial \lambda_-(t,x)}{\partial x}| \leq C|\frac{\partial \lambda_-(\gamma,0)}{\partial \gamma}| \tag{2.57}$$

Noting (2.8), (2.9) and (2.12) we have

$$|\frac{\partial \lambda_-(t,0)}{\partial t}| \leq C(|\frac{\partial}{\partial t}(\frac{2(h \cdot v)}{1+|h|^2})| + |\Lambda'_+(\alpha)||\frac{d\alpha}{dt}|) \tag{2.58}$$

Noting $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\begin{cases} \dfrac{dx(t)}{dt} = \lambda_-(t, x_2(t,\alpha)) \\ t = 0 : \quad x_2(0,\alpha) = \alpha \end{cases} \tag{2.59}$$

Noting $\tilde{C}_2 : x = x_2(t)$ satisfies

$$\begin{cases} \dfrac{dx(t)}{dt} = \lambda_-(t, x_2(t,\alpha)) \\ t = 0 : \quad x_2(0,\alpha) = \alpha \end{cases} \tag{2.59}$$

Therefore

$$|\frac{\partial \lambda_-(t,0)}{\partial t}| \le C(|h'||v| + |h||\frac{\partial v}{\partial t}| + |\Lambda'_+(\alpha)|) \tag{2.60}$$

Noticing

$$v_i(\gamma, 0) = R_i(\gamma, 0) - \lambda_+(\gamma, 0) h_i(\gamma)$$

Then, by the estimate obtained in the previous case we can estimate $|\frac{\partial v}{\partial t}|$, thus

$$|\frac{\partial \lambda_-(t,0)}{\partial t}| \le C(|h'| + |h|N + |f''(\alpha)| + |g'(\alpha)|) \tag{2.61}$$

Therefore

$$|\frac{\partial \lambda_-(t,x)}{\partial x}| \le C(M + N) \tag{2.62}$$

By the same method, we can get

$$|\frac{\partial S_i(t,x)}{\partial x}| \le C(M + N) \tag{2.64}$$

Proof of Theorem 1.1 Under the assumptions of Theorem 1.1, by Lemma 2.1-2.4, on the domain D

$$\|\lambda_\pm\|_1, \|R_i\|_1, \|S_i\|_1 \leq C(M+N+1)$$

$$(\lambda_+(t,x) - \lambda_-(t,x)) \geq b + a$$

Noting (1.24), we can get uniform a priori estimate of C^1 norm of u and v. i.e. system (1.16) have the global C^1 solutions. Then, the system (1.5)-(1.7) have global C^2 solutions.

★**Uniform a priori estimate** For any fixed $T \geq 0$, we introduce

$$W_1(T) = \max_{i=1,\ldots,n} \sup_{0 \leq t \leq T} \{ \int_0^{+\infty} |\frac{\partial \lambda_+}{\partial x}(t,x)|dx, \int_0^{+\infty} |\frac{\partial \lambda_-}{\partial x}(t,x)|dx$$

$$\int_0^{+\infty} |\frac{\partial R_i}{\partial x}(t,x)|dx, \int_0^{+\infty} |\frac{\partial S_i}{\partial x}(t,x)|dx \} \qquad (3.1)$$

$$\tilde{W}_1(T) = \max_{i=1,\ldots,n} \{ \sup_{\tilde{C}_1} \int_{\tilde{C}_1} |\frac{\partial \lambda_+}{\partial x}(t,x)|dt, \sup_{\tilde{C}_2} \int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}(t,x)|dt$$

$$\sup_{\tilde{C}_1} \int_{\tilde{C}_1} |\frac{\partial R_i}{\partial x}(t,x)|dt, \sup_{\tilde{C}_2} \int_{\tilde{C}_2} |\frac{\partial S_i}{\partial x}(t,x)|dt \} \qquad (3.2)$$

where $\tilde{C}_1$ stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ in the domain $[0,T] \times R^+$;
$\tilde{C}_2$ stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ in the domain $[0,T] \times R^+$.

Lemma 3.1 Under the assumptions of Theorem 1.3, there exists a positive constant C only depending on N_0, a, b such that, the following estimates hold:

$$\tilde{W}_1(T), W_1(T) \le C(N_2 + M_2 + M_1 N) \tag{3.3}$$

Proof: Differentiating the system (1.20) with respect to x. We have

$$\begin{cases} \partial_t(\frac{\partial \lambda_-}{\partial x}) + \partial_x(\lambda_+ \frac{\partial \lambda_-}{\partial x}) = 0 \\ \partial_t(\frac{\partial \lambda_+}{\partial x}) + \partial_x(\lambda_- \frac{\partial \lambda_+}{\partial x}) = 0 \\ \partial_t(\frac{\partial R_i}{\partial x}) + \partial_x(\lambda_- \frac{\partial R_i}{\partial x}) = 0 \quad (i = 1, ...n) \\ \partial_t(\frac{\partial S_i}{\partial x}) + \partial_x(\lambda_+ \frac{\partial S_i}{\partial x}) = 0 \quad (i = 1, ...n) \end{cases} \tag{3.4}$$

We rewrite (3.5) as

$$\begin{cases} d|\frac{\partial \lambda_-}{\partial x}|(dx - \lambda_+ dt) = 0 \\ d|\frac{\partial \lambda_+}{\partial x}|(dx - \lambda_- dt) = 0 \\ d|\frac{\partial R_i}{\partial x}|(dx - \lambda_- dt) = 0 \quad (i = 1, ...n) \\ d|\frac{\partial S_i}{\partial x}|(dx - \lambda_+ dt) = 0 \quad (i = 1, ...n) \end{cases} \tag{3.6}$$

In the following we only prove

$$\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt \le C(M_2 + N_2 + M_1 N) \tag{3.16}$$

There are only three possibilities:

Case 1: For any fixed $\alpha \in R^+$, let $\tilde{C}_2$: $x = x_2(t,\alpha)$ stands for any given backward characteristic, passing through the point $A(0,\alpha)$ on the x axis and intersecting $t = T$ at a point P. We draw a forward characteristic $\tilde{C}_1 : x = x_1(t,\beta)$ from P downward and intersects x axis at a point $B(0,\beta)$.
Then, we integrate Equation (3.15) in the region APB to get

$$\int_{\tilde{C}_2} (\lambda_+ - \lambda_-)|\frac{\partial \lambda_-}{\partial x}|(t,x)dt = \int_{\beta}^{\alpha} |\Lambda'_-(x)|dx \tag{3.17}$$

Notice that Lemma 2.1 and (2.2), (3.2), we can get

$$\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt \le CW_1(0) \le CN_2 \tag{3.18}$$

Case 2: For any fixed $\alpha \in R^{+}$, let $\tilde{C}_2$: $x = x_2(t,\alpha)$ stands for any given backward characteristic, passing through the point $A(0,\alpha)$ on the x axis and intersecting $t = T$ at a point P. We draw a backward characteristic $\tilde{C}_1 : x = x_1(t)$ from P downward and intersects t axis at a point $B(\gamma, 0)$.

Then, we integrate Equation (3.15) in the region $PAOB$ to get

$$\int_{\tilde{C}_2} (\lambda_+ - \lambda_-)|\frac{\partial \lambda_-}{\partial x}|(t,x)dt = \int_0^{\gamma} \lambda_+ |\frac{\partial \lambda_-}{\partial x}(t,0)|dt + \int_0^{\alpha} |\Lambda'_-(x)|dx \qquad (3.19)$$

Using the third equation of system (1.20) and Lemma 2.1, we have

$$\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt \le C(\int_0^{\gamma} |\frac{\partial \lambda_-}{\partial t}|(t,0)dt + \int_0^{\alpha} |\Lambda'_-(x)|dx) \qquad (3.20)$$

Then, noting (2.61)

$$\begin{aligned}\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt &\le C(\int_0^{+\infty} |\frac{\partial \lambda_-}{\partial t}|(t,0)dt + \int_0^{+\infty} |\Lambda'_-(x)|dx) \\ &\le C(M_2 + N_2 + M_1 N) \qquad (3.21)\end{aligned}$$

Case 3: For any fixed $\alpha \in R^{+}$, let $\tilde{C}_2$: $x = x_2(t,\alpha)$ stands for any given backward characteristic, passing through the point $A(0,\alpha)$ on the x axis and intersecting t axis at a point $B(\gamma, 0)$.

Then, we integrate equation (3.15) in the region AOB to get

$$\int_0^{\gamma} (\lambda_+ - \lambda_-)|\frac{\partial \lambda_-}{\partial x}|(t,x)dt = \int_0^{\alpha} |\Lambda'_-(x)|dx + \int_0^{\gamma} \lambda_+|\frac{\partial \lambda_-}{\partial x}|(t,0)dt \quad (3.22)$$

Similarly, we can get

$$\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt \le C(\int_0^{\alpha} |\Lambda'_-(x)|dx + \int_0^{\gamma} |\frac{\partial \lambda_-}{\partial t}|(t,0)dt)$$

Then

$$\int_{\tilde{C}_2} |\frac{\partial \lambda_-}{\partial x}|(t,x)dt \le C(M_2 + N_2 + M_1 N) \quad (3.23)$$

Lemma 3.2 Under the assumptions of Theorem 1.3, we have

$$\{\int_{L_1}(1-\lambda_+)(t,x)dt, \int_{L_2}(1+\lambda_-(t,x))dt,$$

$$\int_{L_1}|R_i(t,x)|dt, \int_{L_2}|S_i(t,x)|dt\} \le C(N_1+M_1) \tag{3.31}$$

$$\{\int_{\tilde{C}_1}(1-\lambda_+(t,x))dt, \int_{\tilde{C}_2}(1+\lambda_-(t,x))dt,$$

$$\int_{\tilde{C}_1}|R_i(t,x)|dt, \int_{\tilde{C}_2}|S_i(t,x)|dt\} \le C(N_1+M_1) \tag{3.32}$$

where $\tilde{C}_1$ stands for any given forward characteristic $\frac{dx}{dt} = \lambda_+$ in the domain $[0,T]\times R^+$;
$\tilde{C}_2$ stands for any given backward characteristic $\frac{dx}{dt} = \lambda_-$ in the domain $[0,T]\times R^+$; L_1 stands for any given radial that has the slope 1 in the domain $[0,T]\times R^+$; L_2 stands for any given radial that has the slope -1 in the domain $[0,T]\times R^+$.

★ The proof of Theorem 1.3

Let

$$\frac{D}{D_1 t} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \tag{4.1}$$

Obviously,

$$\frac{D}{D_1 t} = T_1 + (1 - \lambda_+)\frac{\partial}{\partial x} \tag{4.2}$$

Thus, noting system (1.20)

$$\frac{D\lambda_-}{D_1 t} = T_1 \lambda_- + (1 - \lambda_+)\frac{\partial \lambda_-}{\partial x} \tag{4.3}$$

In the following we consider Equation (4.3), i.e.

$$\frac{D\lambda_-}{D_1 t} = (1 - \lambda_+)\frac{\partial \lambda_-}{\partial x} \tag{4.4}$$

For any fixed point $(t, x) \in D$, define $\xi = x - t$

Case 1: $\xi \geq 0$, it follows Equation (4.4) that

$$\lambda_-(t,x) = \lambda_-(t,\xi+t) = \lambda_-(0,\xi) + \int_0^t (1-\lambda_+)\frac{\partial \lambda_-}{\partial x}(s,\xi+s)ds \tag{4.5}$$

By (2.65) and Lemma 3.2, we have

$$\begin{aligned} &|\int_0^t (1-\lambda_+)\frac{\partial \lambda_-}{\partial x}(s,\xi+s)|ds \\ &\leq W_\infty(\infty)\int_0^{+\infty} |1-\lambda_+|(s,\xi+s)dx \\ &\leq C(M+N)(M_1+N_1) \end{aligned} \tag{4.6}$$

This implies that the integral $\int_0^t (1-\lambda_+)\frac{\partial \lambda_-}{\partial x}(s,\xi+s)ds$ converges uniformly for $\xi \in R^+$, On the other hand, noting that all functions in the right-hand side in Equation (4.5) are continuous with respect to ξ. Then, we observe that there exists a unique function $\tilde{\psi}(\xi) \in C^0(R^+)$ such that

$$\lambda_-(t,x) \longrightarrow \tilde{\psi}(x-t) \quad t \longrightarrow \infty \tag{4.7}$$

Case 2: $\xi \leq 0$, it follows Equation (4.5) that

$$\lambda_-(t,x) = \lambda_-(t,\xi+t) = \lambda_-(-\xi,0) + \int_{-\xi}^{t} (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds \quad (4.8)$$

By (2.65) and Lemma 3.2, we can get

$$\begin{aligned} &\int_{-\xi}^{t} (1-\lambda_+)\frac{\partial\lambda_-}{\partial x}(s,\xi+s)ds \\ &\leq W_\infty(\infty)\int_{-\xi}^{t}(1-\lambda_+)(s,\xi+s)ds \\ &\leq C(M+N)(M_1+N_1) \end{aligned} \quad (4.9)$$

Then,we obtain that there exists a unique function $\bar{\psi}(\xi) \in C^0(R^-)$ such that

$$\lambda_-(t,x) \longrightarrow \bar{\psi}(x-t) \quad t \longrightarrow +\infty \quad (4.10)$$

Case 3: When $\xi \longrightarrow 0$, notice that above cases we can get

$$\tilde{\psi}(\xi) \longrightarrow \tilde{\psi}(0) \text{ and } \bar{\psi}(\xi) \longrightarrow \bar{\psi}(0) \tag{4.11}$$

Moreover

$$\tilde{\psi}(0) = \bar{\psi}(0) \tag{4.12}$$

We define

$$\psi(\xi) = \begin{cases} \tilde{\psi}(\xi), & \xi \in R^+; \\ \bar{\psi}(-\xi), & \xi \in R^-; \end{cases}$$

Hence from above we have proved the following lemma

Lemma 4.1 There exists a unique function $\psi(x-t) \in C^0(R)$, such that

$$\lambda_-(t,x) \longrightarrow \psi(x-t) \quad t \longrightarrow +\infty \tag{4.13}$$

Remark 4.1 In the same way, we can obtain that there exists a unique function $\psi_i(x-t) \in C^0(R)$ such that

$$S_i(t,x) \longrightarrow \psi_i(x-t) \quad t \longrightarrow +\infty \quad i = 1, ..., n \tag{4.14}$$

Lemma 4.2 When $t \longrightarrow +\infty$, we have

$$\lambda_+(t,x) \longrightarrow 1 \tag{4.15}$$

$$R_i(t,x) \longrightarrow 0 \qquad i = 1, ..., n \tag{4.16}$$

uniformly for all $x \geq 0$.

Noting (1.24),

$$\lim_{t\longrightarrow\infty} u_i(t,x) = \lim_{t\longrightarrow\infty} \frac{R_i(t,x)-S_i(t,x)}{\lambda_+(t,x)-\lambda_-(t,x)} = \lim_{t\longrightarrow\infty} \frac{-\psi_i(x-t)}{1-\psi(x-t)} \tag{4.17}$$

$$\begin{aligned}\lim_{t\longrightarrow\infty} v_i(t,x) &= \lim_{t\longrightarrow\infty} \frac{\lambda_+ S_i(t,x)-\lambda_- R_i(t,x)}{\lambda_+(t,x)-\lambda_-(t,x)}\\ &= \lim_{t\longrightarrow\infty} \frac{\psi_i(x-t)}{1-\psi(x-t)}\end{aligned} \tag{4.18}$$

Then, when $x \geq 0$, we can get

$$\lim_{t\longrightarrow\infty} u_i(t,x) = \frac{-\psi_i(x-t)}{1-\psi(x-t)} \doteq \Phi_i(x-t) \quad i=1,...,n \tag{4.19}$$

$$\lim_{t\longrightarrow\infty} v_i(t,x) = \frac{\psi_i(x-t)}{1-\psi(x-t)} \doteq -\Phi_i(x-t) \quad i=1,...,n \tag{4.20}$$

We next prove that $\Phi_i(\xi) \in C^1(R)$. Noting $\psi_i(\xi), \psi(\xi) \in C^0(R)$, we need to show that $d(\psi_i(\xi))/d\xi, d(\psi(\xi))/d\xi \in C^0(R)$. It suffices to show that $\psi(\xi), \psi_i(\xi) \in C^1(R)$.

In the following we only prove $\psi(\xi) \in C^1(R)$

Lemma 4.3 Under the assumptions of Theorem 1.3, the limit

$$\lim_{t\longrightarrow\infty}\frac{\partial\lambda_-}{\partial x}(t,x_1(t,\beta))\doteq\psi^*(\beta) \tag{4.21}$$

exists and is continuous. Moreover

$$|\psi^*(\beta)|\le C(M+N)(N_2+M_2+M_1N) \tag{4.22}$$

Lemma 4.4 The limit

$$\lim_{t\longrightarrow\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t)$$

exists and is continuous with respect to ξ.

Lemma 4.5

$$\frac{d\psi(\xi)}{d\xi}=\lim_{t\longrightarrow\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t). \tag{4.65}$$

Remark 4.3 In the similar way, we also can prove

$$\lim_{t\longrightarrow\infty}\frac{\partial S_i}{\partial x}(t,\xi+t)=\frac{d\psi_i(\xi)}{d\xi} \tag{4.69}$$

Lemma 4.6

$$\lim_{t\longrightarrow\infty}\frac{\partial\lambda_-}{\partial x}(t,\xi+t)=\psi^*(\vartheta(\xi)) \tag{4.70}$$

is continuous in R. Moreover

$$\frac{d\psi(\xi)}{d\xi}=\psi^*(\vartheta(\xi)) \tag{4.71}$$

Remark 4.4 By the same method, we obtain that $\frac{\partial S_i}{\partial x}(t,\xi+t)$ have the similar conclusion. Moreover

$$\frac{d\psi_i(\xi)}{d\xi}=\psi_i^*(\vartheta(\xi)) \tag{4.75}$$

Thank You